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Exact solutions to nonlinear equations with quadratic nonlinearity

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Abstract

Based on the concepts of the generalized conditional symmetry and of the invariant subspace, a constructive way to derive nonlinear equations with quadratic nonlinearity that admit exact solutions as well as many new examples is presented. This type of solution, found by Galaktionov, describes the blow-up phenomena. Examples of exact solutions that do not correspond to the invariant subspace are also presented and explained by the nonlinear generalized conditional symmetries.

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There exist equations with quadratic nonlinearity that admit exact solutions describing the blow-up phenomena. The first one, found by Galaktionov [1], is the following:

$$u_t = u_{xx} + u_x^2 - k^2u^2 \tag{1}$$

which admits the solution

$$\begin{aligned} &u = g_1(t)\mathrm{e}^{kx} + g_2(t)\mathrm{e}^{-kx} + g_3(t) \\ &g_1' = k^2g_1(1-2g_3) \qquad g_2' = k^2g_2(1-2g_3) \qquad g_3' = -k^2(g_3^2 + 4g_1g_2). \end{aligned} \tag{2}$$

Other equations that admit similar exact solutions include [2, 3]

$$u_t = uu_{xx} - \tfrac{3}{4}u_x^2 - \tfrac{1}{4}k^2u^2 \tag{3}$$

and

$$u_t = uu_{xx} - \tfrac{2}{3}u_x^2 + \tfrac{1}{6}kuu_x - \tfrac{1}{6}k^2u^2. \tag{4}$$

There are several ways to explain the existence of such exact solutions. One such way is the concept of generalized condition symmetry (GCS) [4]. $\sigma(u)$ is a GCS of the equation $u_t = K(u)$, iff

$$K'\sigma - \sigma'K = F(u,\sigma) \qquad F(u,0) = 0$$

where $F(u,\sigma)$ is a differentiable function of $u, u_x, u_{xx}, \ldots$ and of $\sigma, \sigma_x, \sigma_{xx}, \ldots$. For each GCS $\sigma(u)$, there is a solution of the equation $u_t = K(u)$ that also satisfies $\sigma(u) = 0$. It is easy to verify that $\sigma = u_{xxx} - k^2 u_x$ is a GCS of equation (1), and (2) is the solution associated with this GCS. Another way to explain such solutions is the concept of the invariant subspace [2]. A linear space V of differential functions is an invariant subspace of a nonlinear operator K if $K[V] \subseteq V$. If V is an invariant subspace of K, then there exists a $u \in V$ such that $K(u) = 0$. It is easy to see that $V = \{e^{kx}, e^{-kx}, 1\}$ is an invariant subspace of $K(u) = -u_t + u_{xx} + u_x^2 - k^2 u^2$.

The GCS $\sigma = u_{xxx} - k^2 u_x$ of equation (1), as well as the GCSs of equations (3) and (4), are all linear GCSs. In fact, there is a close relation between the linear GCS and the invariant subspace.

Lemma 1. *Φu is a GCS of equation $u_t = K[u]$, iff $V = \{u : \Phi u = 0\}$ is an invariant subspace of $K[u]$.*

Proof. $(\Phi u)' K[u]|_{\Phi u=0} = \Phi K[u]|_{\Phi u=0} = \Phi K[u]|_{u\in V}$.

In this paper, we present a constructive way to derive equations with quadratic nonlinearity that admit exact solutions similar to (2).

For convenience, for a set of numbers L, we denote the operator $\Phi_L = \Pi_{l\in L}(\partial_x - l)$.

Lemma 2. *Suppose L is a set of numbers. If there exist $A, B \subseteq L$ such that $A + B \subseteq L$, then $\Phi_L u = 0$ is an invariant subspace of $F(\partial_x)u + (\Phi_{L-A}u)(\Phi_{L-B}u)$, where F is a polynomial with constant coefficients.*

Proposition 1. *Suppose L is a set of numbers. If there exist $A, B \subseteq L$ such that $A + B \subseteq L$, then $\Phi_L u$ is a GCS of the equation $u_t = F(\partial_x)u + (\Phi_{L-A}u)(\Phi_{L-B}u)$, where F is a polynomial with constant coefficients.*

In the following we show a few examples based on proposition 1.

(1) Let $L = (0, k, -k)$, $A = (0, k)$, $B = (0, -k)$, $F(\partial_x) = \partial_x^2$, then we know that $\Phi_L u = \partial_x(\partial_x - k)(\partial_x + k)u = u_{xxx} - k^2 u_x$ is a GCS of the equation

$$u_t = u_{xx} + [(\partial_x - k)u][(\partial_x + k)u]$$

which is the equation (1).

(2) Let $L = (0, k_1, -k_1, k_2)$, $A = (0, k_1)$, $B = (0, -k_1)$, $F(\partial_x) = 0$, then we know that $\Phi_L u = u_{xxxx} - k_2 u_{xxx} - k_1^2 u_{xx} + k_1^2 k_2 u_x$ is a GCS of the equation

$$u_t = u_{xx}^2 - 2k_2 u_x u_{xx} + (-k_1^2 + k_2^2)u_x^2 + 2k_1^2 k_2 u u_x - k_1^2 k_2^2 u^2.$$

The associated solution is

$$u = g_1(t)e^{k_1 x} + g_2(t)e^{-k_1 x} + g_3(t)e^{k_2 x} + g_4(t) \qquad g_1' = 2k_1^2 k_2(k_1 - k_2)g_1 g_4$$
$$g_2' = -2k_1^2 k_2(k_1 + k_2)g_2 g_4 \qquad g_3' = 0 \qquad g_4' = 4k_1^2(k_1^2 - k_2^2)g_1 g_2 - k_1^2 k_2^2 g_4^2.$$

(3) Let $L = (0, k_1, k_2, k_1 + k_2)$, $A = (0, k_1)$, $B = (0, k_2)$, $F(\partial_x) = 0$, then we know that $\Phi_L u = \partial_x(\partial_x - k_1)(\partial_x - k_2)(\partial_x - k_1 - k_2)u$ is a GCS of the equation

$$u_t = [(\partial_x - k_1)(\partial_x - k_1 - k_2)u][(\partial_x - k_2)(\partial_x - k_1 - k_2)u].$$

(4) Let $L = (0, k_1, k_2, \frac{1}{2}k_1)$, $A = B = (0, \frac{1}{2}k_1)$, $F(\partial_x) = 0$, then we know that $\Phi_L u = \partial_x(\partial_x - k_1)(\partial_x - k_2)(\partial_x - \frac{1}{2}k_1)u$ is a GCS of the equation

$$u_t = [(\partial_x - k_1)(\partial_x - k_2)u]^2.$$

(5) Let $L = (0, k, \frac{1}{2}k, -\frac{1}{2}k)$, $F(\partial_x) = 0$, then $A = (0, k, \frac{1}{2}k)$, $B = (0, -\frac{1}{2}k)$, or $A = (0, \frac{1}{2}k, -\frac{1}{2}k)$, $B = (0, \frac{1}{2}k)$; therefore we know that $\Phi_L u = \partial_x(\partial_x - k)(\partial_x - \frac{1}{2}k)(\partial_x + \frac{1}{2}k)u$ is a GCS of the equation

$$u_t = [(\partial_x + \tfrac{1}{2}k)u][(\partial_x - k)(\partial_x - \tfrac{1}{2}k)u] - [(\partial_x - k)u][(\partial_x - k)(\partial_x + \tfrac{1}{2}k)u]$$

which is equation (4).

(6) Let $L = (0, k, -k, \frac{1}{2}k)$, $F(\partial_x) = 0$, then $A = (0, k)$, $B = (0, -k)$, or $A = B = (0, \frac{1}{2}k)$; therefore we know that $\Phi_L u = \partial_x(\partial_x - k)(\partial_x + k)(\partial_x - \frac{1}{2}k)u$ is a GCS of the equation

$$u_t = u_x u_{xx} - 2kuu_{xx} + \tfrac{3}{4}ku_x^2 - k^2uu_x + \tfrac{5}{4}k^3u^2.$$

(7) Let $L = (0, k, -k, \frac{1}{2}k, -\frac{1}{2}k)$, $F(\partial_x) = 0$, then $A = (0, k, \frac{1}{2}k)$, $B = (0, -k, -\frac{1}{2}k)$, or $A = B = (0, \frac{1}{2}k, -\frac{1}{2}k)$; therefore $\Phi_L u = \partial_x(\partial_x^2 - k^2)(\partial_x^2 - \frac{1}{4}k^2)u$ is a GCS of the equation

$$u_t = [(\partial_x + k)(\partial_x + \tfrac{1}{2}k)u][(\partial_x - k)(\partial_x - \tfrac{1}{2}k)u] - [(\partial_x + k)(\partial_x - k)u]^2$$

which is equation (3).

(8) Let $L = (0, k_1, -k_1, k_2, -k_2)$, $F(\partial_x) = 0$, then $A = (0, k_1)$, $B = (0, -k_1)$, or $A = (0, k_2)$, $B = (0, -k_2)$; therefore we know that $\Phi_L u = \partial_x(\partial_x^2 - k_1^2)(\partial_x^2 - k_2^2)u$ is a GCS of the following two equations:

$$u_t = 2u_x u_{xxx} - u_{xx}^2 - (k_1^2 + k_2^2)u_x^2 + k_1^2k_2^2u^2$$
$$u_t = u_{xxx}^2 - (k_1^2 + k_2^2)u_{xx}^2 + k_1^2k_2^2(2uu_{xx} - u_x^2).$$

(9) As a generalization of the last example, we have the following equation:

$$u_t = Au\sum_{k=0}^{n} a_k\partial_x^{2k}u + B\sum_{k=0}^{n} a_k\sum_{l=0}^{2k}(-1)^l(\partial_x^l u)(\partial_x^{2k-l}u) \tag{5}$$

where A and B are arbitrary constants, which admits the solution

$$u = \sum_{i=1}^{n}\left(g_{2i-1}(t)e^{k_i x} + g_{2i}(t)e^{-k_i x}\right) + g_{2n+1}(t) \tag{6}$$

with k_i^2 $(1 \leqslant i \leqslant n)$ the roots of the polynomial $\sum_0^n a_k x^k$, and $g_i(t)$ $(1 \leqslant i \leqslant 2n+1)$ satisfy the following system of equations:

$$\begin{aligned} &\partial_t g_{2i-1} = Aa_0 g_{2i-1}g_{2n+1} \qquad \partial_t g_{2i} = Aa_0 g_{2i}g_{2n+1} \qquad (1 \leqslant i \leqslant n)\\ &\partial_t g_{2n+1} = (A+B)a_0 g_{2n+1}^2 + 2B\sum_{i=1}^{n}\left[\sum_{k=0}^{n} a_k(2k+1)k_i^{2k}\right]g_{2i-1}g_i. \end{aligned} \tag{7}$$

This solution corresponds to the linear GCS $\sigma = \sum_0^n a_k\partial_x^{2k+1}u$.

Nonlinear equations with quadratic nonlinearity may admit exact solutions associated with nonlinear GCSs. Although these solutions may not correspond to invariant subspaces, our approach may still be useful in these cases. One such example is the equations admitting the following solution:

$$u = g_1(t)e^{k_1x} + g_2(t)e^{k_2x} + g_3(t)e^{\frac{1}{2}(k_1+k_2)x} + g_4(t). \tag{8}$$

Consider

$$V = \{e^{k_1x}, e^{k_2x}, e^{\frac{1}{2}(k_1+k_2)x}, 1\} = \{u : \partial_x(\partial_x - k_1)(\partial_x - k_2)(\partial_x - \tfrac{1}{2}(k_1+k_2))u = 0\}.$$

Since

$$(\partial_x - k_1)(\partial_x - \tfrac{1}{2}(k_1+k_2))u \in \{e^{k_2x}, 1\} \qquad (\partial_x - k_2)(\partial_x - \tfrac{1}{2}(k_1+k_2))u \in \{e^{k_1x}, 1\}$$

we know that

$$K = [(\partial_x - k_1)(\partial_x - \tfrac{1}{2}(k_1+k_2))u][(\partial_x - k_2)(\partial_x - \tfrac{1}{2}(k_1+k_2))u] \in \tilde{V}$$

where

$$\tilde{V} = \{e^{k_1x}, e^{k_2x}, e^{(k_1+k_2)x}, e^{\frac{1}{2}(k_1+k_2)x}, 1\}.$$

Since

$$(\partial_x - k_1)(\partial_x - k_2)u \in \{e^{\frac{1}{2}(k_1+k_2)x}, 1\}$$

we know that

$$K = [(\partial_x - k_1)(\partial_x - k_2)u]^2 \in \tilde{V}.$$

The combination of these two K yields

$$K = u_{xx}^2 - 2(k_1+k_2)u_x u_{xx} + (k_1^2 + 3k_1k_2 + k_2^2)u_x^2 - k_1k_2(k_1+k_2)uu_x$$

which also belongs to $\tilde{V}$. In general, $K[u]$ with u given in (8) may not belong to V. But when the g_i satisfy a certain condition, it is possible that $K[u] \in V$. In such case, u becomes a solution of $u_t = K$. We can find such a condition directly as follows. For u given in (8),

$$K[u] = -k_1^2k_2(k_1+k_2)g_1g_4e^{k_1x} - k_1k_2^2(k_1+k_2)g_2g_4e^{k_2x} - \tfrac{1}{2}k_1k_2(k_1+k_2)^2g_3g_4e^{\frac{1}{2}(k_1+k_2)x} \\ +(k_1-k_2)^2\left[\tfrac{1}{16}(k_1+k_2)^2g_3^2 - k_1k_2g_1g_2\right]e^{(k_1+k_2)x}.$$

Therefore if $(k_1+k_2)^2g_3^2 = 16k_1k_2g_1g_2$, then (8) is a solution of $u_t = K[u]$. The system of equations that the g_i have to satisfy is

$$g_1' = -k_1^2k_2(k_1+k_2)g_1g_4 \qquad g_2' = -k_1k_2^2(k_1+k_2)g_2g_4$$
$$g_3' = -\tfrac{1}{2}k_1k_2(k_1+k_2)^2g_3g_4 \qquad g_4' = 0.$$

Solving this system, we find the solution

$$u = 1 + e^{k_1z+A_1} + e^{k_2z+A_2} + \frac{4\sqrt{k_1k_2}}{k_1+k_2}e^{\frac{1}{2}(k_1+k_2)z+\frac{1}{2}(A_1+A_2)} \qquad z = x - (k_1+k_2)k_1k_2t$$

of the equation

$$u_t = u_{xx}^2 - 2(k_1+k_2)u_xu_{xx} + (k_1^2+3k_1k_2+k_2^2)u_x^2 - k_1k_2(k_1+k_2)uu_x. \tag{9}$$

We also notice that for u given in (8), $K[u]$ in general contains the term $e^{\frac{1}{2}(k_1+k_2)x}$, therefore $K[u]$ is not an invariant subspace for the u given in (8). Therefore Galaktionov's approach does not work here, nor does this solution correspond to linear GCS. However, actually this solution corresponds to a quadratic GCS. This is because this solution satisfies a linear equation $u_t = -k_1k_2(k_1+k_2)u_x$. From this relation and equation (9), we know that this solution corresponds to the quadratic GCS:

$$\sigma = u_{xx}^2 - 2(k_1+k_2)u_xu_{xx} + (k_1^2+3k_1k_2+k_2^2)u_x^2 - k_1k_2(k_1+k_2)uu_x + k_1k_2(k_1+k_2)u_x.$$

Similar to the case of bilinear equations [5], exact solutions of equations with quadratic nonlinearity can also be sought directly. We consider the equation

$$\sum_{i,j}\gamma_{ij}\partial_x^i\partial_t^j u = \sum_{i,j}\alpha_{ij}(\partial_x^i u)(\partial_x^j u). \tag{10}$$

Without loss of generality, we assume that $\alpha_{ij} = \alpha_{ji}$.

Introducing

$$F(x,y) = \sum_{i,j}\gamma_{ij}x^iy^j \qquad P(x,y) = \sum_{i,j}\alpha_{ij}x^iy^j \tag{11}$$

then equation (10) can be written as

$$F(\partial_x, \partial_t)u(x,t) = P(\partial_x, \partial_{x'})u(x,t)u(x',t)|_{x'=x}. \tag{12}$$

We search for solutions in the following form:

$$u(x,t) = \sum_p g_p(t)e^{k_px}. \tag{13}$$

Substituting this into equation (10), we obtain

$$\sum_{p}\left[\sum_{i,j}\gamma_{ij}k_p^i g_q^{(j)}(t)\right]\mathrm{e}^{k_p x}=\sum_{p,q}\left[\sum_{i,j}\alpha_{ij}k_p^i k_q^j g_p(t)g_q(t)\right]\mathrm{e}^{(k_p+k_q)x} \tag{14}$$

i.e.

$$\sum_{p}F(k_p,\partial_t)g_p(t)\mathrm{e}^{k_p x}=\sum_{p,q}P(k_p,k_q)g_p(t)g_q(t)\mathrm{e}^{(k_p+k_q)x}. \tag{15}$$

One necessary condition for equation (15) to be valid is that for any p, q, either $P(k_p, k_q) = 0$ or $k_p + k_q = k_r$ or $k_p + k_q = k_r + k_s$ for some r, s.

Now we consider some special forms of (13). Firstly we search for solutions of equation (10) in the form of (8). Equating all coefficients in (15), we know that P satisfies

$$P(k_1,k_1)=P(k_2,k_2)=P(k_1,\tfrac{1}{2}(k_1+k_2))=P(k_2,\tfrac{1}{2}(k_1+k_2))=0. \tag{16}$$

$g_1(t), g_2(t), g_3(t), g_4(t)$ satisfy the following system of equations:

$$\begin{array}{ll} F(k,\partial_t)g_1=2P(k_1,0)g_1g_4 & F(k_2,\partial_t)g_2=2P(k_2,0)g_2g_4 \\ F(\tfrac{1}{2}(k_1+k_2),\partial_t)g_3=2P(\tfrac{1}{2}(k_1+k_2),0)g_3g_4 & F(0,\partial_t)g_4=P(0,0)g_4^2 \end{array} \tag{17}$$

and $g_1(t), g_2(t), g_3(t)$ also hold the following algebraic relation:

$$2P(k_1,k_2)g_1g_2+P(\tfrac{1}{2}(k_1+k_2),\tfrac{1}{2}(k_1+k_2))g_3^2=0. \tag{18}$$

If $F(x, y) = y$, then equations (17) become

$$\begin{array}{ll} g_1'=2P(k_1,0)g_1g_4 & g_2'=2P(k_2,0)g_2g_4 \\ g_3'=2P(\tfrac{1}{2}(k_1+k_2),0)g_3g_4 & g_4'=P(0,0)g_4^2. \end{array} \tag{19}$$

Equations (18) and (19) imply that $P(k_1, 0) + P(k_2, 0) = 2P(\frac{1}{2}(k_1 + k_2), 0)$. Therefore we know that if P satisfies

$$\begin{array}{l} P(k_1,k_1)=P(k_2,k_2)=P(k_1,\tfrac{1}{2}(k_1+k_2))=P(k_2,\tfrac{1}{2}(k_1+k_2))=0 \\ P(k_1,0)+P(k_2,0)=2P(\tfrac{1}{2}(k_1+k_2),0) \end{array} \tag{20}$$

then equation

$$u_t=\sum_{i,j}\alpha_{ij}(\partial_x^i u)(\partial_x^j u) \tag{21}$$

admits the following solutions:

$$u=-\frac{1}{P(0,0)t}+t^{-\frac{2P(k_1,0)}{P(0,0)}}\mathrm{e}^{k_1x+A_1}+t^{-\frac{2P(k_2,0)}{P(0,0)}}\mathrm{e}^{k_2x+A_2} +\left[-\frac{2P(k_1,k_2)}{P(\frac{1}{2}(k_1+k_2),\frac{1}{2}(k_1+k_2))}\right]^{\frac{1}{2}}t^{-\frac{P(k_1,0)+P(k_2,0)}{P(0,0)}}\mathrm{e}^{\frac{1}{2}(k_1+k_2)x+\frac{1}{2}(A_1+A_2)}$$

if $P(0, 0) \neq 0$, and

$$u=1+\mathrm{e}^{k_1x+2P(k_1,0)t+A_1}+\mathrm{e}^{k_2x+2P(k_2,0)t+A_2} +\left[-\frac{2P(k_1,k_2)}{P(\frac{1}{2}(k_1+k_2),\frac{1}{2}(k_1+k_2))}\right]^{\frac{1}{2}}\mathrm{e}^{\frac{1}{2}(k_1+k_2)x+(P(k_1,0)+P(k_2,0))t+\frac{1}{2}(A_1+A_2)}$$

if $P(0, 0) = 0$, where A_1, A_2 are arbitrary constants.

It is easy to verify that

$$P(x,y)=x^2y^2-(k_1+k_2)xy(x+y)+(k_1^2+3k_1k_2+k_2^2)xy-\tfrac{1}{2}k_1k_2(k_1+k_2)(x+y)$$

satisfies the conditions (20). The corresponding evolution equation is exactly equation (9), and we also recover its exact solution.

Similarly, the search for solutions in the form of

$$u = g_1(t)\mathrm{e}^{kx} + g_2(t)\mathrm{e}^{-kx} + g_3(t)\mathrm{e}^{\frac{1}{3}kx} + g_4(t)\mathrm{e}^{-\frac{1}{3}kx} + g_5(t)$$

reveals that equation

$$u_t = u_{xx}^2 + au_x^2 + \tfrac{3}{16}a^2u^2$$

admits the solution

$$u = g_1(t)\mathrm{e}^{kx} + g_2(t)\mathrm{e}^{-kx} + 9g_1^{2/3}(t)g_2^{1/3}(t)\mathrm{e}^{\frac{1}{3}kx} + 9g_1^{1/3}(t)g_2^{2/3}(t)\mathrm{e}^{-\frac{1}{3}kx} + g_5(t)$$

$$k = \sqrt{-\tfrac{3}{4}a} \qquad g_1' = \tfrac{3}{8}a^2g_1g_5 \qquad g_2' = \tfrac{3}{8}a^2g_2g_5 \qquad g_5' = \tfrac{3}{16}a^2(256g_1g_2 + g_5^2).$$

It is easy to verify that $V = \{\mathrm{e}^{kx}, \mathrm{e}^{-kx}, \mathrm{e}^{kx/3}, \mathrm{e}^{-kx/3}, 1\}$ is not an invariant subspace of $K[u] = u_{xx}^2 + au_x^2 + \frac{3}{16}a^2u^2$. Therefore, once again, we obtain a solution that corresponds to nonlinear GCS.

In summary, this paper has provided a new constructive method (proposition 1) for systematically obtaining quadratic nonlinear evolution equations with exact solutions, based on the concepts of invariant subspaces and generalized conditional symmetries. Although some examples of such equations have been known for quite a while, our approach is much simpler and more efficient than the existing ones. Many equations admitting exact solutions are presented in this paper, which are as simple as the known equations while they have been unknown until now. The exact solutions derived based on proposition 1 correspond to invariant subspaces, as well as to the linear generalized conditional symmetries. We further showed, through examples, that the idea behinds proposition 1 could be useful to even find exact solutions that do not correspond to invariant subspaces. In general, such solutions should correspond to nonlinear generalized conditional symmetries. We also showed a direct approach to the search for such solutions. Both of these approaches are experimental and the conditions of the existence of such solutions deserve more investigation in future studies.

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